EUROPEAN ORGANISATION FOR NUCLEAR RESEARCH

CERN – A&B DEPARTMENT

AB-Note-2008-043 ABP

# OPTIMIZING COST AND MINIMIZING ENERGY LOSS IN THE RECIRCULATING RACE-TRACK DESIGN OF THE LHEC ELECTRON LINAC

**J. Skrabacz**

**Abstract**

The objective of this project is to propose an optimal design of a recirculating electron linac for a future LHC-based e-p collider—the LHeC [1, 2]. Primary considerations are the cost, structure, shape, and size of the recirculating track, the optimal number of revolutions through which the e-beam should be accelerated, and radiative energy loss in the bends. Secondary considerations are transverse emittance growth due to radiation, the number of dipoles needed in order to maintain an upper bound on the emittance growth, the average length of such dipoles, and the maximum bending dipole field needed to recirculate the beam. These effects will be studied macroscopically with respect to the overall structure, in that smaller effects related to machine optics of the lattice structure will be neglected. The scope of the optimization problem is, in essence, a "first order" insight into optimal dimensions, centered on minimizing the most important parameter—cost.

*CERN, Geneva, Switzerland*
*August 2008*

# Optimizing Cost and Minimizing Energy Loss in the Recirculating Race-Track Design of the LHeC Electron Linac

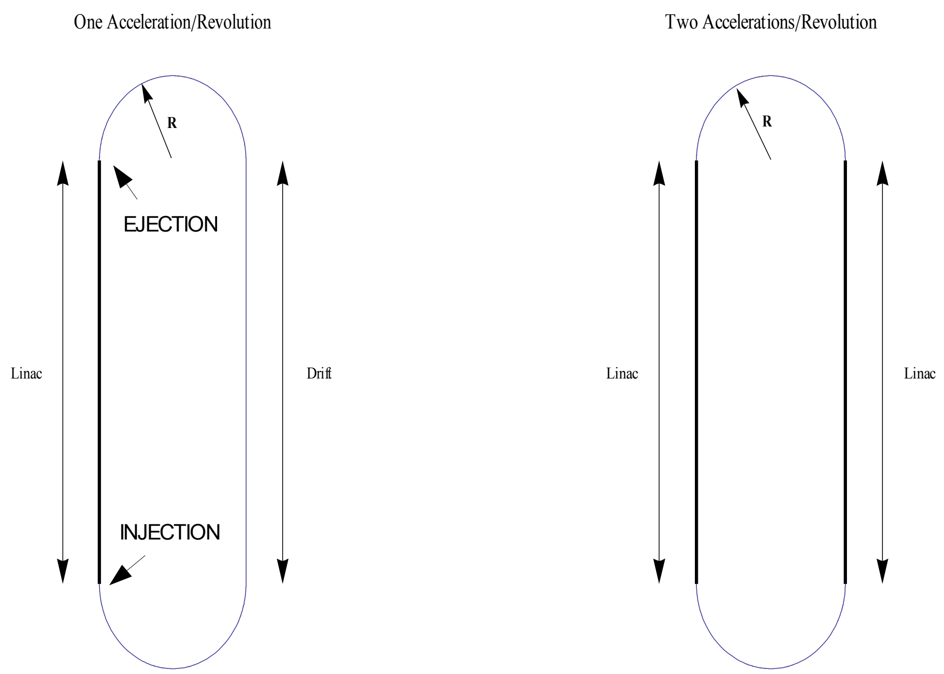
**Jake Skrabacz**
**U.M. CERN REU Aug 2008**

## I. Introduction

The objective of this project is to propose an optimal design of a recirculating electron linac for a future LHC-based e-p collider—the LHeC [1, 2]. Primary considerations are the cost, structure, shape, and size of the recirculating track, the optimal number of revolutions through which the $e^-$ beam should be accelerated, and radiative energy loss in the bends. Secondary considerations are transverse emittance growth due to radiation, the number of dipoles needed in order to maintain an upper bound on the emittance growth, the average length of such dipoles, and the maximum bending dipole field needed to recirculate the beam. These effects will be studied macroscopically with respect to the overall structure, in that smaller effects related to machine optics of the lattice structure will be neglected. The scope of the optimization problem is, in essence, a "first order" insight into optimal dimensions, centered on minimizing the most important parameter—cost.

## II. LHeC

Although it would not be the site of the first ever e-p collisions, a future LHeC would be the first ever site of electrons colliding with protons at energies as high as 7TeV—extending the discovery reach of the LHC. This project is a prefix to any new highest-energy e-p physics. It focuses on finding an optimal structure for the $e^-$ linac—subject to the constraint that a specific target energy is reached. The primary shape studied for the recirculating linac is the "race-track" design. Structurally, this project looks at two variations of it.




The "race-track" design limits the structure of the linac+recirculation to 4 parameters: length (of the linac and/or drift sections), radius (of the bending track), boolean (which is given a value of TRUE, or 1, for a doubly accelerating structure and FALSE, or 0, for the singly accelerating structure), and the number of revolutions through which the $e^-$ beam will be accelerated and recirculated.

Another design that will be briefly addressed later is the 5-parameter "ball-field" design, which calls for a slightly more complicated algorithm and will be compared to the race-track design. .

## III. Analysis

In the machine, there are two sources of energy change for the $e^-$ beam: acceleration by the linac and energy loss to synchrotron radiation in the bends. This model will neglect smaller sources of radiative energy loss in the linac, in the quadrupoles, from wake-fields, etc.

Radiative Energy Loss and Linac Energy Gain

In the bends, the synchrotron radiation power as a function of the electron's energy is given by:

$$P_{syn} = \frac{e^2 c}{6\pi\epsilon_0}\frac{1}{(m_0 c^2)^4}\frac{E^4}{R^2}$$

*R is the radius of our bend.

This leads to a first order differential equation, which can be solved analytically to give the total change in energy when an $e^-$ beam is bent by θ radians.

$$\frac{1}{E^4}dE = \frac{-ac}{2\pi}\frac{1}{R^2}d\,t = -\frac{a}{2\pi}\frac{1}{R}d\theta \quad \text{where } a = \frac{e^2}{3\epsilon_0}\frac{1}{(m_0 c^2)^4} \quad \text{and } (**)\ dt \simeq \frac{R}{c}d\theta$$

(**) *dealing with highly relativistic particles, we can assume v ≈ c*

This yields our total energy change to radiation loss:

$$\Delta E_{Rad}(E_0, R, \theta) = -E_0\{1 - \sqrt[3]{\frac{r}{3aE_0^3 + r}}\}\ \text{ where } r = \frac{2\pi R}{\theta}$$

*$E_0$ is the energy at the start of the bend.

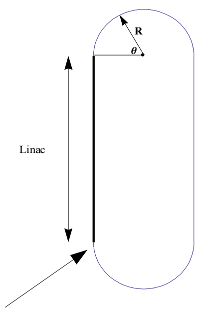
All energy gain takes place along the length of the linac in the form:

$$\Delta E_{Linac}(L) = \frac{dE}{dx}[GeV/m]\cdot L$$

*dE/dx is our energy gradient (the energy gain per unit length of the linac).

Recursive Energy Function

From these two sources of energy change—given the injection energy, the type of structure (boolean = 0,1), and the total number of revolutions (N) through which the $e^-$ beam passes, one can construct a recursive formula for the ejection energy of the particles:




$$E_{target} = E\,(E_{inj}, R, L, dE/dx, N, bool) = E_N(E_{N-1}(...E_1(E_0)...)) + \frac{dE}{dx}L$$

$$\text{i.e. } E_i = E_{i-1} + \frac{dE}{dx}L + \Delta E_{rad}(E_{i-1} + dE/dx\cdot L, R, 2\pi) \text{ for the bool} = 0 \text{ structure}$$

*$E_i$ refers to the energy at the injection point at the start of the i-th revolution so $E_{inj} = E_0$.

Cost

Assuming a constant cost per unit length for the three types of tracks—linac, drift, bending—we can trivially formulate the total cost of the design from basic geometries.

$$\text{Total Cost} = 2\pi\ RN(\$bend) + (1 + \delta_{1,bool})L(\$linac) + \delta_{0,bool}L(\$drift)$$

*\$linac, \$drift, \$bending obviously refer to the cost/m for each type of track.
*$\delta$ is the Kronecker Delta, introduced to account for the two different structure options.
*A factor of N—the number of revolutions—must be introduced to the bending term of the cost, since each revolution comes with a different energy, thus requiring a higher dipole field for each loop of the bend (unless an FFAG-like optics with extremely large momentum acceptance could be employed).
Consequently, we assume that for each bend, there is cost-wise a “new” track.

Effective Cost

The goal of this project is to find the optimal radius, length, type of structure, and number of revolutions to minimize the cost formula given above, while achieving the target energy for the machine; however, this optimization problem does not constrain energy loss from radiation. Relatively high energy losses ($E_{loss}/E_{target}$ > 5%) would give an inefficient design with a high operating cost. It is therefore appropriate to introduce an effective cost, with the aid of the dummy weight parameter λ.

$$Effective\ Cost = Total\ Cost + \lambda \times |\Delta E_{rad}(total)|$$

* λ has units of $1,000,000/GeV, literally giving a cost per unit of energy loss.

Emittance Growth

Beam emittance in the transverse plane is significantly affected by radiation loss. For emittance in the x (radial) direction we have:

$$\epsilon_x = \frac{\sigma_x^2}{\beta_x} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{x x'}^2}$$

After some derivation this yields [3]:

$$\Delta\gamma\epsilon_x \simeq (4\times 10^{-8}\ m^2\ GeV^{-6})\cdot E^6 \sum \frac{\ell_i \langle H \rangle_i}{R_i^3}$$

*$l_i$ gives the length of the i-th dipole in our bend

Assuming there is a homogeneous distribution of D identical dipoles per bend, we find [4]:

$$F = \frac{R^2 \langle H \rangle}{\ell^3} \simeq 0.1,\ \ell = \frac{2\pi R}{D} \implies \Delta\gamma\epsilon_x \sim \frac{\left(6.2\times 10^{-6}\ m^2\ GeV^{-6}\right)\cdot E^6}{R\, D^3}$$

Assuming the same number of dipoles for each revolution's lattice structure, after N revolutions we find:

$$(\Delta\gamma\epsilon_x)_{total} \sim \frac{\left(6.2\times 10^{-6}\ m^2\ GeV^{-6}\right)}{R\, D^3} \sum_{i=0}^{N-1} (E_i + dE/dx\cdot L)^6 \quad for\ bool = 0\ structure$$

$$(\Delta\gamma\epsilon_x)_{total} \sim \frac{\left(3.1\times 10^{-6}\ m^2\ GeV^{-6}\right)}{R\, D^3} \sum_{i=0}^{2N-1} (E_i + dE/dx\cdot L)^6 \quad for\ bool = 1\ structure$$

** There is a factor of 1/2 for the emittance growth for the doubly-accelerating structure. There are twice as many accelerations in a given revolution (as indicated by the upper index of the sum),so the i-th component emittance growth is only valid for a semi-circle (πR)–summed over 1/2 the number of dipoles per revolution.*

Setting an upper limit on this value [($\Delta\gamma\varepsilon_x$)< 100μm] will yield a minimum number of dipoles needed for each revolution of the total circumference, while $2\pi R/D_{min}$ will give the maximum average length needed for each one.

Dipole Bending Field

A Taylor series approximation for the bending field (assuming dependence only on x) yields:

$$\frac{e}{p} B(x) = \frac{e}{p} B_0 + \frac{e}{p}\frac{dB}{dx} x + \frac{1}{2!}\frac{e}{p}\frac{d^2B}{dx^2} x^2 + \frac{1}{3!}\frac{e}{p}\frac{d^3B}{dx^3} x^3 + ...$$

In the dipole approximation, only the first term is considered:

$$\frac{e}{p} B_{dip}(x) = \frac{e}{p} B_0 = \frac{1}{R}$$

For highly relativistic particles:

$$E \simeq pc \implies B(E) \simeq \frac{E}{ecR} = \frac{10}{3}\frac{E\,[GeV]}{R} \implies B_{max} = B(E_{target} - \frac{dE}{dx} L)$$

*In finding the optimal effective cost parameters for our design, it is necessary to check what $B_{max}$ is, since bending fields over 2 T require superconducting magnets—where the cost of dipoles and the bending track would jump. Such an outcome would render the algorithm ineffective, since it would have required a greater value for $bend.

**IV. Computational Algorithm**

The core of this project's algorithm, is the optimization of the effective cost—yielding a dual effect of minimizing cost and, to the variable extent of the weight parameter λ, radiative energy loss. This optimization problem (*again neglecting specific effects of the machine's optics or beam properties)* calls for 11 variables: the effective cost, which needs to be minimized, is a function of 10 variables, while the constraint parameter—the target energy—is another variable:

| Variable Type | Variable(s) | Symbol(s) | Units |
|---|---|---|---|
| Size | Radius (bend), length (linac/drift) | R, L | m, km |
| Energy | injection, target energies | $E_i$, $E_t$ | MeV, GeV |
| Linac Energy Gain | energy gradient | dE/dx | MeV/m |
| Structure | boolean (singly, doubly accelerating) | Bool | 0, or 1 |
| Design | number of revolutions | N | integer |
| Cost | cost of linac, drift, bending | \$linac, \$drift, \$bend | \$1,000/m |
| Weight Parameter | effective cost per unit energy loss | λ | \$1,000,000/GeV |

Having already constructed formulas for total effective cost and the final energy, the optimal dimensions for our machine are the ones that minimize the total effective cost subject to the constraint that the final energy of such machine is exactly equal to the target energy of the machine. In order to simplify and expedite the computational optimization process, it is helpful to narrow the function to two variables— radius and length. This is made possible by either setting a constant value for all other parameters, or specifying an incremented range over which the optimization algorithm will be run for multiple trials. Here are the parameters used:

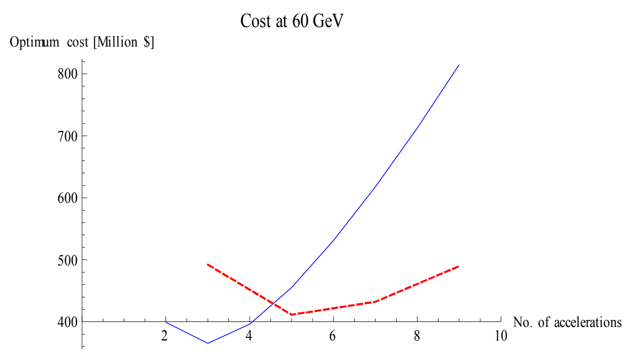
| Parameter | Value/Range |
|---|---|
| $E_i$ | 500 GeV |
| $E_t$ | {20, 40, 60, 80, 100, 120} GeV |
| dE/dx | 15 MeV/m |
| N | trials from 1 to 8 |
| bool | {0, 1} |
| \$linac | \$160,000/m |
| \$drift | \$15,000/m |
| \$bend | \$50,000/m |
| λ | {0, 1, 10, 100, 1000, 10000} \$Mill./GeV |

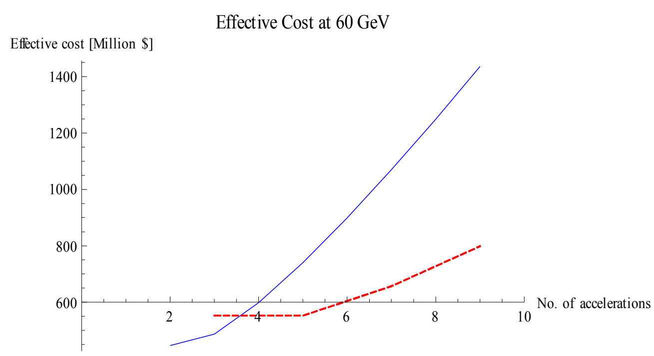
We now have a optimization problem for a function of two variables → Find R* and L* on the curve E (R, L) = $E_t$, such that $(\text{Effective Cost})_{\text{Min}}$ = Effective Cost $(R, L)|_{R = R^*, L = L^*}$. Using R* and L*, the formulas constructed in the analysis will yield values for cost, effective cost, circumference, energy loss, the number of dipoles per bend needed to keep an upper bound on emittance growth, the average length per dipole, and the maximum bending field required to bend the beam at it's greatest energy.

**V. Sample Results**

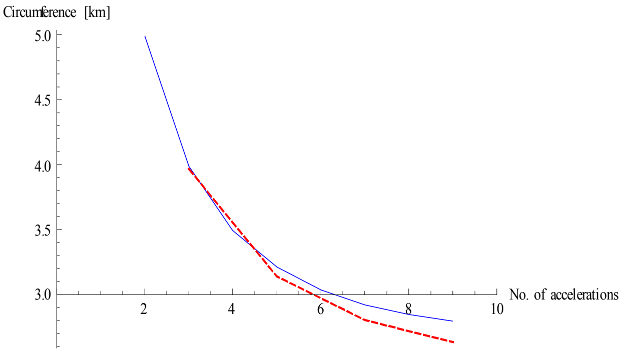
1. In these following samples, λ is fixed at \$100 Million/GeV. This first example is the results gathered for a target energy of 60 GeV. Red represents the doubly-accelerating structure, and blue is the singly-accelerating structure. The results span over the range of possible revolutions, optimizing effective cost and giving the minimal optimal design—namely the structure type and number of revolutions. The first number in each row of the table refers to the number of accelerations for the design.

$$n_{acc}(\text{bool} = 0) = N + 1 \text{ and } n_{acc}(\text{bool} = 1) = 2N + 1$$

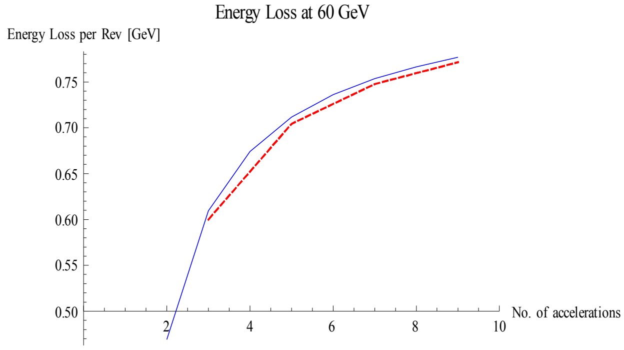


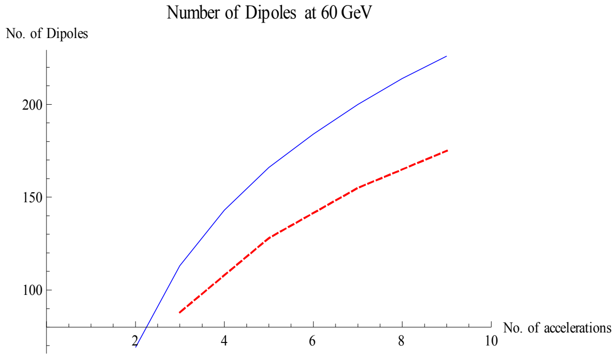
Circumference at 60 GeV



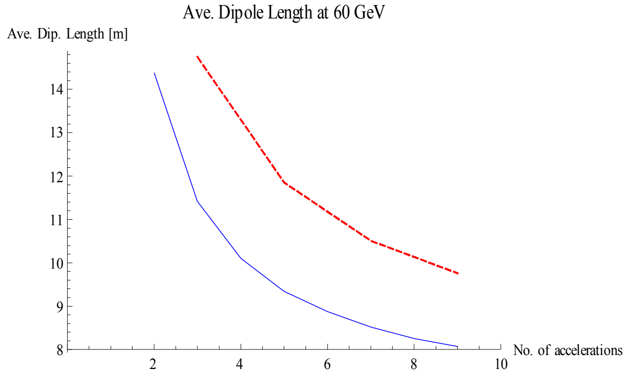

Energy Loss at 60 GeV



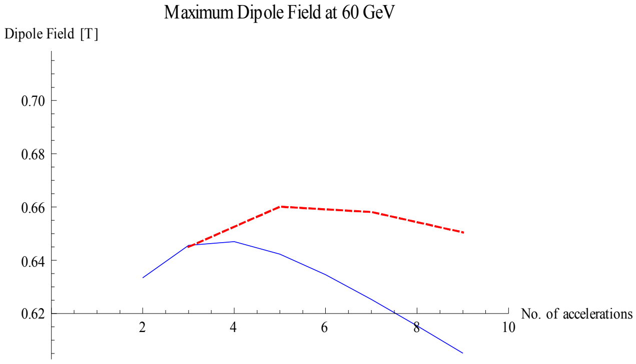

Number of Dipoles at 60 GeV




Ave. Dipole Length at 60 GeV




Maximum Dipole Field at 60 GeV

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Minimum cost at 60 GeV is $$365.086 Million at 2 revs, for the 1 acc/rev structure.

Minimum effective cost at 60 GeV is $$446.308 Million at 1 revs, for the 1 acc/rev structure.

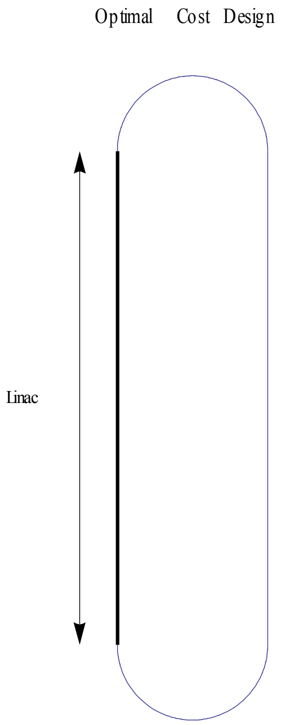
Results for singly accelerating structure.
```

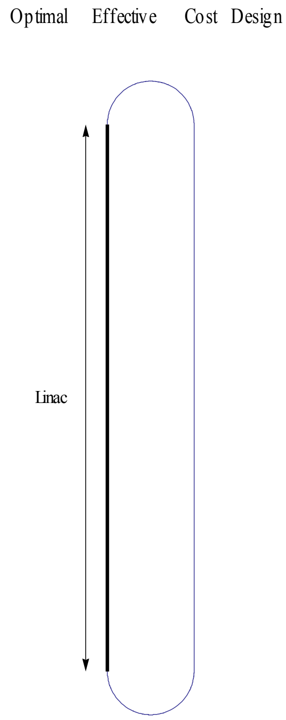
| | Rad. [m] | Length [km] | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 157.982 | 1.99864 | 4.98991 | 399.393 | 446.308 | 0.469146 | 69 | 14.3859 | 0.633415 |
| 3 | 205.304 | 1.34909 | 3.98814 | 365.086 | 486.98 | 0.609466 | 113 | 11.4156 | 0.645608 |
| 4 | 229.869 | 1.0252 | 3.49471 | 396.056 | 598.268 | 0.674039 | 143 | 10.1 | 0.647065 |
| 5 | 246.673 | 0.831149 | 3.21219 | 455.43 | 740.05 | 0.711549 | 166 | 9.33671 | 0.642317 |
| 6 | 259.866 | 0.701898 | 3.03658 | 531.029 | 899.107 | 0.736156 | 184 | 8.87384 | 0.634577 |
| 7 | 271.077 | 0.609634 | 2.9225 | 617.654 | 1069.81 | 0.753594 | 200 | 8.51614 | 0.625351 |
| 8 | 281.078 | 0.540471 | 2.84701 | 712.705 | 1249.36 | 0.766643 | 214 | 8.25264 | 0.615404 |
| 9 | 290.281 | 0.4867 | 2.79729 | 814.728 | 1436.18 | 0.776817 | 226 | 8.07031 | 0.605155 |

```
Results for doubly accelerating structure.
```

| | Rad. [m] | Length [km] | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|---|---|
| 2 | -- | -- | -- | -- | -- | -- | -- | -- | -- |
| 3 | 206.541 | 1.33534 | 3.96841 | 492.194 | 552.202 | 0.600083 | 88 | 14.747 | 0.645069 |
| 4 | -- | -- | -- | -- | -- | -- | -- | -- | -- |
| 5 | 241.451 | 0.811986 | 3.14105 | 411.543 | 552.442 | 0.70449 | 128 | 11.8522 | 0.660179 |
| 6 | -- | -- | -- | -- | -- | -- | -- | -- | -- |
| 7 | 259.207 | 0.587936 | 2.80452 | 432.437 | 656.76 | 0.747745 | 155 | 10.5074 | 0.658173 |
| 8 | -- | -- | -- | -- | -- | -- | -- | -- | -- |
| 9 | 271.827 | 0.463526 | 2.63499 | 489.916 | 798.515 | 0.771497 | 175 | 9.75964 | 0.650502 |

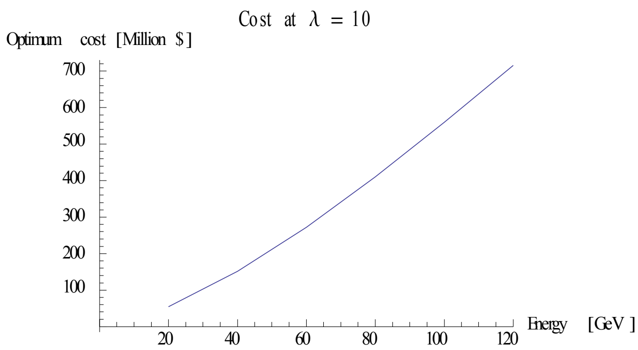
This yields the following plots of the optimal designs (to scale intrinsically but not with respect to each other).

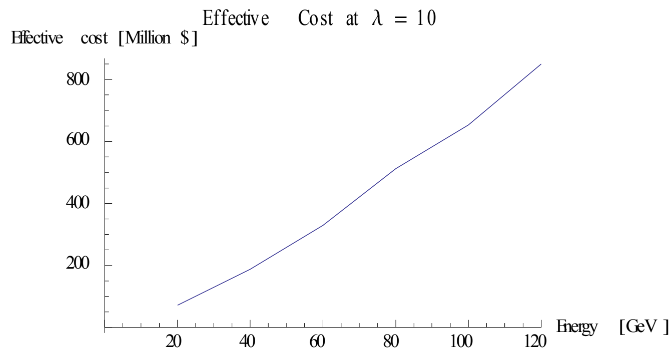


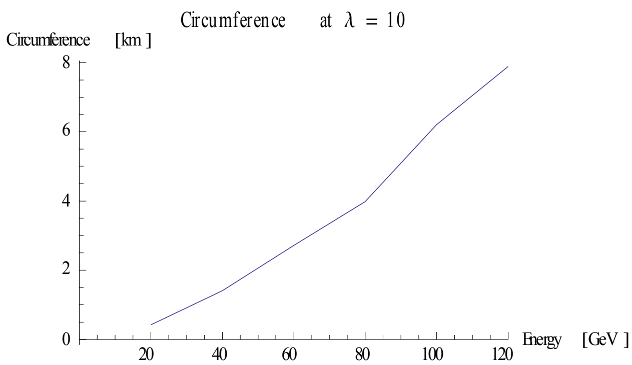



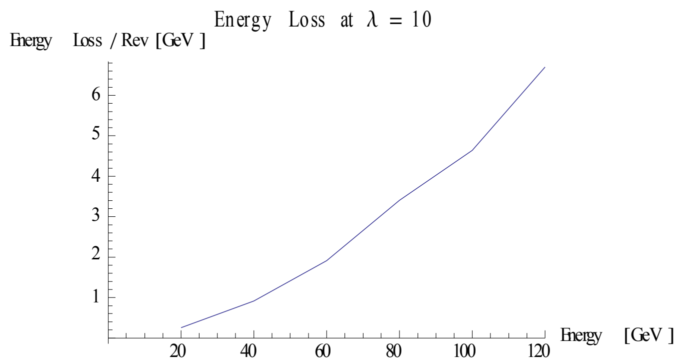

2. The following results summarize the optimal cost and effective cost parameters and structure for each target energy across our energy range for $\lambda$ = $10 Million/GeV.
**On the maximum dipole field plot, the superconducting threshold (~ 2T ) is mapped for reference.*

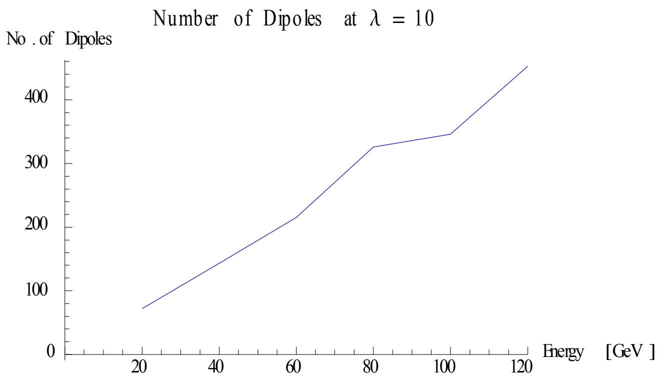
# Results For Optimal COST Structure

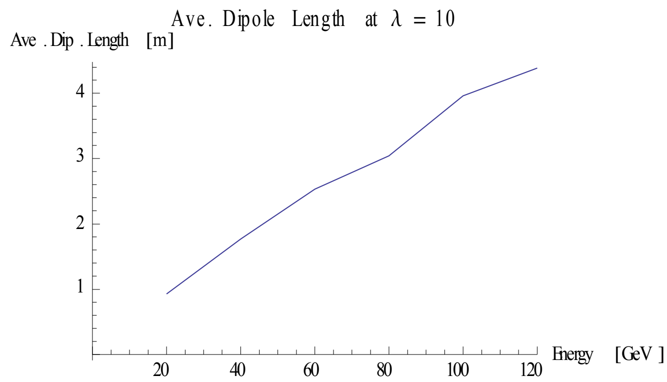


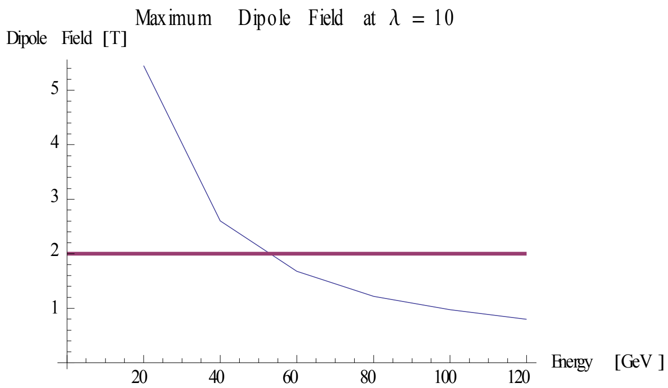





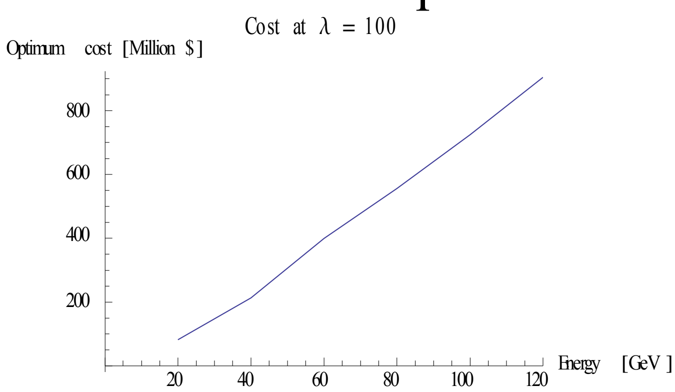



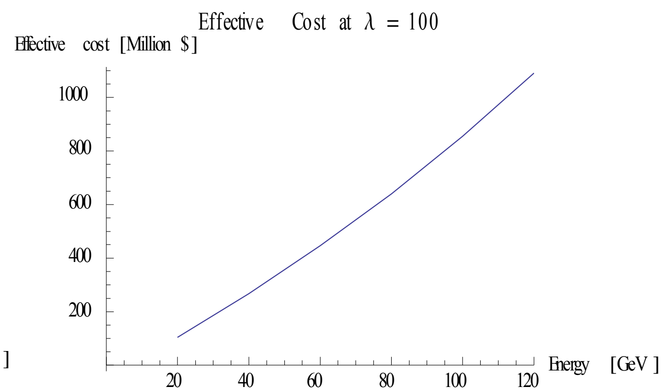



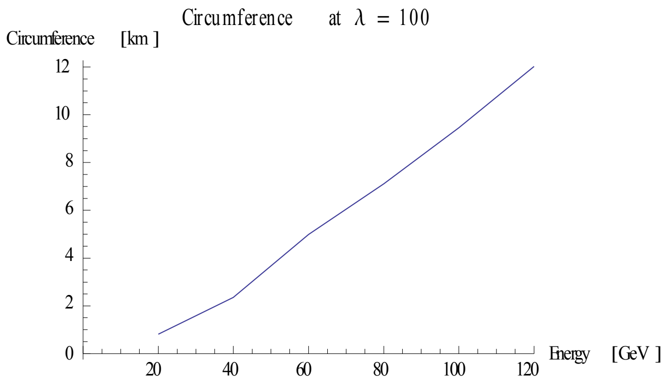

| | Acc/Rev | No. Revs |
|---|---|---|
| E = 20 GeV | 1 | 4 |
| E = 40 GeV | 1 | 2 |
| E = 60 GeV | 1 | 2 |
| E = 80 GeV | 1 | 2 |
| E = 100 GeV | 1 | 1 |
| E = 120 GeV | 1 | 1 |

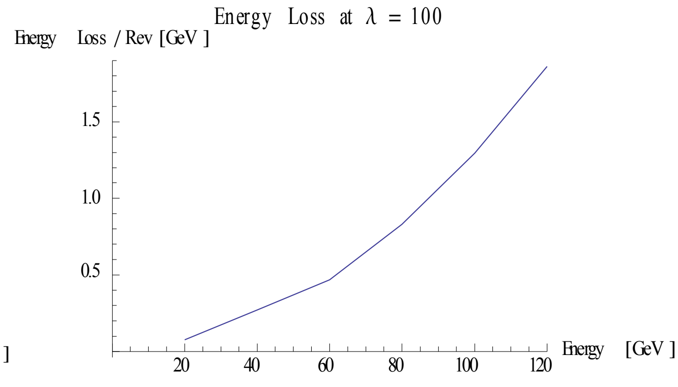
| | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|
| 20 GeV | 0.69591 | 79.7504 | 111.802 | 0.0801298 | 38 | 4.41129 | 2.00381 |
| 40 GeV | 2.35204 | 212.963 | 267.4 | 0.272184 | 66 | 8.67786 | 0.974729 |
| 60 GeV | 3.98814 | 365.086 | 486.98 | 0.609466 | 113 | 11.4156 | 0.645608 |
| 80 GeV | 5.92974 | 547.612 | 763.824 | 1.08106 | 168 | 13.695 | 0.480484 |
| 100 GeV | 9.45627 | 724.775 | 854.239 | 1.29464 | 136 | 20.1272 | 0.379565 |
| 120 GeV | 12.0237 | 904.563 | 1090.7 | 1.86135 | 174 | 22.607 | 0.315864 |

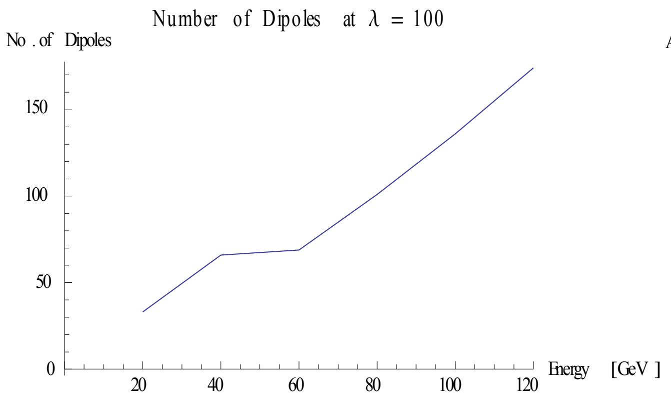
# Results For Optimal EFFECTIVE COST Structure

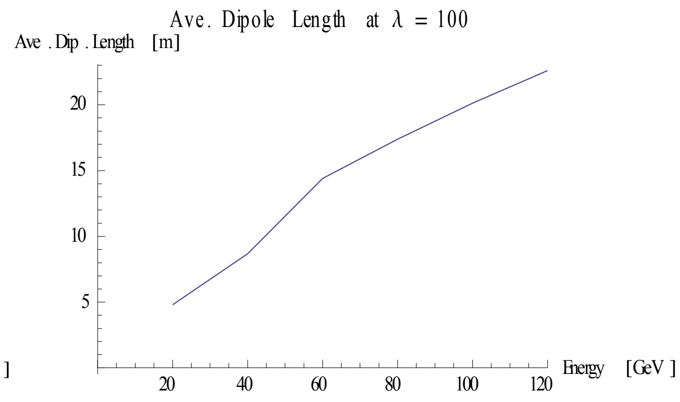


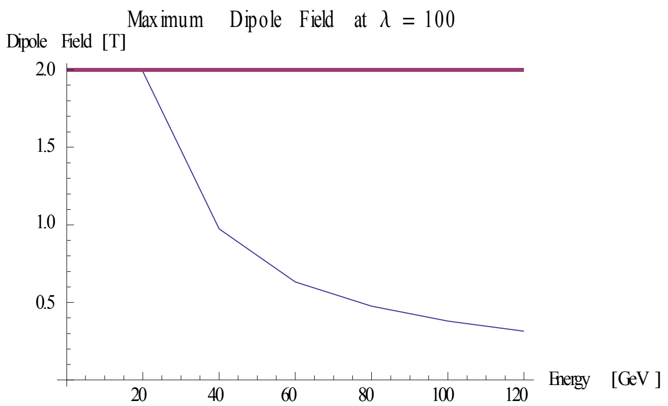






| | Acc/Rev | No. Revs |
|---|---|---|
| E = 20 GeV | 1 | 3 |
| E = 40 GeV | 1 | 2 |
| E = 60 GeV | 1 | 1 |
| E = 80 GeV | 1 | 1 |
| E = 100 GeV | 1 | 1 |
| E = 120 GeV | 1 | 1 |

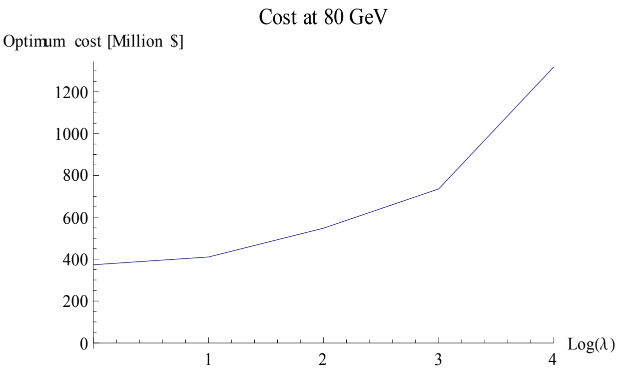
3. Another approach is to look at a fixed energy, and plot the values of the optimal desig parameters as a function of λ. The following have a fixed target energy of 80 GeV, mapping the optimal parameters as a function of Log(λ)–in order to gain insight into an appropriate order of magnitude for λ in the model.

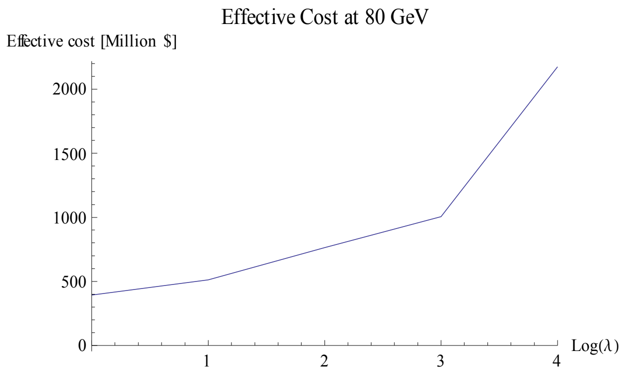
.

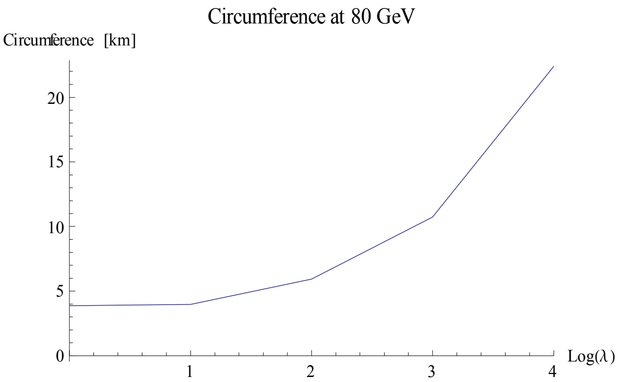
# Results For Optimal COST Structure

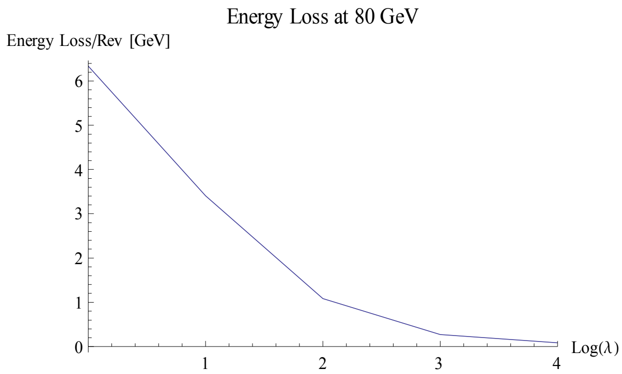
| | Acc/Rev | No. Revs |
|---|---|---|
| λ=1 | 1 | 3 |
| λ=10 | 1 | 3 |
| λ=100 | 1 | 2 |
| λ=1000 | 1 | 1 |
| λ=10000 | 1 | 1 |

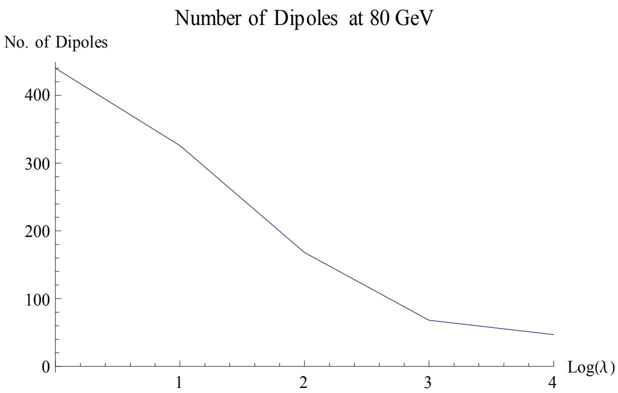
| | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|
| λ=1 | 3.86056 | 373.894 | 392.895 | 6.33375 | 440 | 1.31254 | 2.00828 |
| λ=10 | 3.98128 | 410.312 | 512.438 | 3.40418 | 326 | 3.04049 | 1.21654 |
| λ=100 | 5.92974 | 547.612 | 763.824 | 1.08106 | 168 | 13.695 | 0.480484 |
| λ=1000 | 10.7359 | 736.195 | 1005.58 | 0.269381 | 68 | 79.6859 | 0.155072 |
| λ=10000 | 22.3963 | 1318.76 | 2172.83 | 0.0854074 | 47 | 363.645 | 0.0492767 |

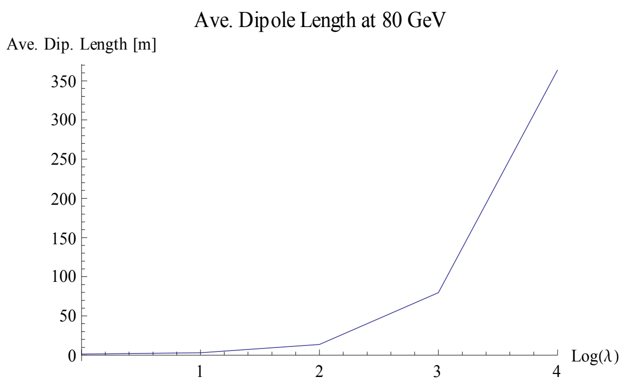

Cost at 80 GeV
Optimum cost [Million $]
Log(λ)

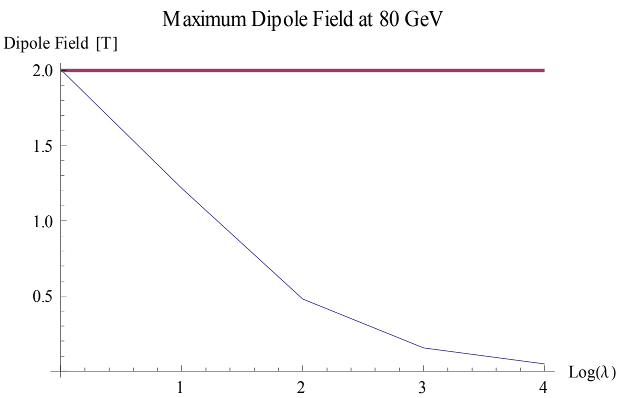

Effective Cost at 80 GeV
Effective cost [Million $]
Log(λ)


Circumference at 80 GeV
Circumference [km]
Log(λ)


Energy Loss at 80 GeV
Energy Loss/Rev [GeV]
Log(λ)

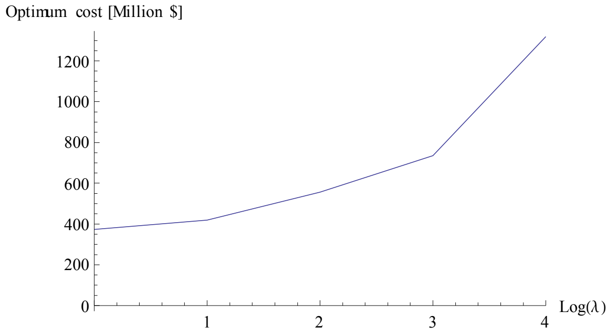

Number of Dipoles at 80 GeV
No. of Dipoles
Log(λ)


Ave. Dipole Length at 80 GeV
Ave. Dip. Length [m]
Log(λ)

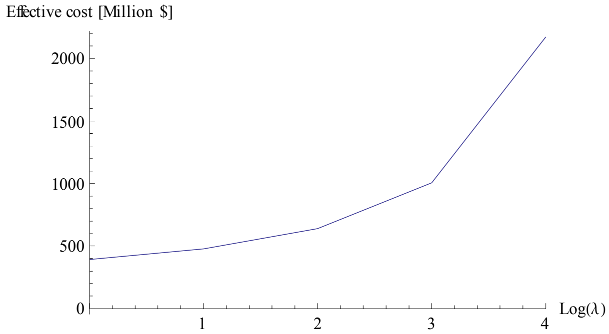

Maximum Dipole Field at 80 GeV
Dipole Field [T]
Log(λ)

# Results For Optimal EFFECTIVE COST Structure

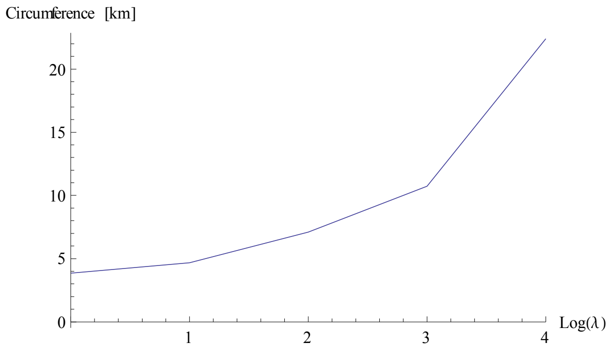
Cost at 80 GeV



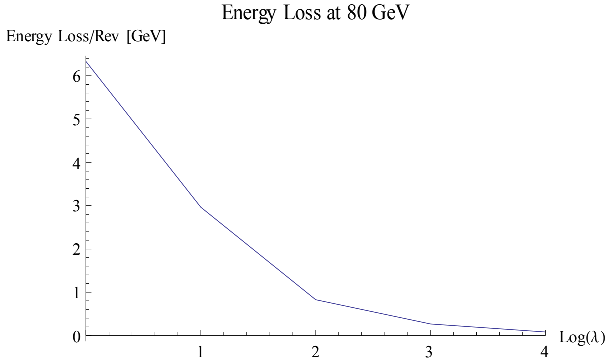

Effective Cost at 80 GeV



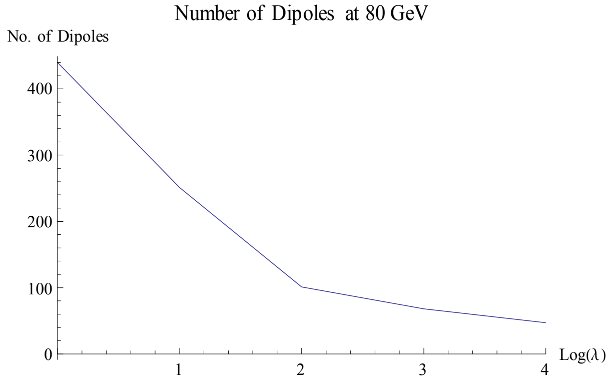

Circumference at 80 GeV



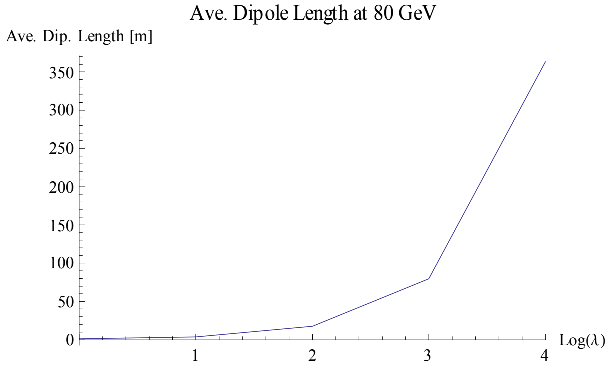

Energy Loss at 80 GeV



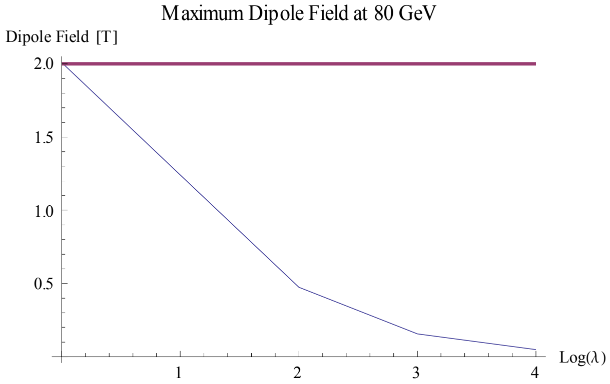

Number of Dipoles at 80 GeV




Ave. Dipole Length at 80 GeV




Maximum Dipole Field at 80 GeV




| | Acc/Rev | No. Revs |
|---|---|---|
| λ=1 | 1 | 3 |
| λ=10 | 1 | 2 |
| λ=100 | 1 | 1 |
| λ=1000 | 1 | 1 |
| λ=10000 | 1 | 1 |

| | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|
| λ=1 | 3.86056 | 373.894 | 392.895 | 6.33375 | 440 | 1.31254 | 2.00828 |
| λ=10 | 4.66643 | 419.184 | 478.548 | 2.96821 | 251 | 3.46496 | 1.2408 |
| λ=100 | 7.11154 | 556.379 | 639.438 | 0.830591 | 101 | 17.3944 | 0.474945 |
| λ=1000 | 10.7359 | 736.195 | 1005.58 | 0.269381 | 68 | 79.6859 | 0.155072 |
| λ=10000 | 22.3963 | 1318.76 | 2172.83 | 0.0854074 | 47 | 363.645 | 0.0492767 |

**VI. Limitations of Model**

1. The model assumes a constant cost of bending track. In reality, the cost of a bending dipole magnet increases with the dipole strength—which goes like 1/R. Therefore, smaller turns require stronger magnets, and would increase the bending track cost. For all domains of the target energy and λ, where the maximum dipole field was below the superconducting region (< 2T), the model provides a good approximation—as there is minimal fluctuation in dipole cost for non-superconducting magnets, and a constant bending track cost is more than reasonable for the aim of this project. For all domains where the model calls for dipole strengths in the superconducting region, the model is ineffective—as the cost of dipoles jumps, requiring a higher value for the $bend variable and then a rerun of the new algorithm.
2. The model was run with the number of accelerations ranging from 2 to 9—corresponding to 1-8 revolutions for the singly accelerating structure and 1-4 revolutions for the doubly accelerating structure. There was no "magical" reason for 8 as the upper limit of the revolution range; it is merely the last value for which Mathematica could evaluate the optimization problem and its resulting parameters in a reasonable amount of time. For energy and λ values that returned 8, or 4 as the optimal number of revolutions for the singly, or doubly accelerating structures, the minimum found is not necessarily a global minimum, since the algorithm computes the minimum of the specified range. Fortunately, this only occurs for cases where λ = 0 and λ = 1 and $E_t$ = 20GeV. Any λ = 0 results are not very useful, since they put no constraint on the amount of energy loss—contrary to one of the primary objectives of the model. The λ = 1 result, too, almost neglects energy loss, while 20 GeV marks the lower bound of the range of target energies looked at. This case lies outside the range of interest for any realistic machine.
3. This model looks mostly at construction cost and, to a limited extent, any operating cost associated with high amounts of radiative energy loss. It does not consider operating cost associated with instrumentation, maintenance, or repair. The objective of the model is to merely provide a starting point in considerations of the dimensions of the machine. Now it's results should be scrutinized in future, modified models that expand the complexity of the machine by introducing a detailed optical structure, aspects of beam dynamics, and more operating cost considerations.

**VII. Conclusions**

Across the range of energy values and weight parameters studied, the results unanimously give a singly accelerating structure as the optimal structure for both minimum cost and minimum effective cost. The following table summarizes the optimal number of revolutions for each (λ, $E_t$) pair studied.

**Optimal Cost Structure (OCS)**

| λ / $E_t$ | 20 | 40 | 60 | 80 | 100 | 120 |
|---|---|---|---|---|---|---|
| **0** | 8 | 6 | 4 | 3 | 3 | 3 |
| **1** | 8 | 5 | 4 | 3 | 3 | 2 |
| **10** | 7 | 4 | 3 | 3 | 2 | 2 |
| **100** | 4 | 2 | 2 | 2 | 1 | 1 |
| **1000** | 2 | 1 | 1 | 1 | 1 | 1 |
| **10000** | 1 | 1 | 1 | 1 | 1 | 1 |

**Optimal Effective Cost Structure (OES)**

| λ / $E_t$ | 20 | 40 | 60 | 80 | 100 | 120 |
|---|---|---|---|---|---|---|
| **0** | 8 | 6 | 4 | 3 | 3 | 3 |
| **1** | 7 | 5 | 4 | 3 | 3 | 2 |
| **10** | 5 | 3 | 2 | 2 | 2 | 1 |
| **100** | 3 | 2 | 1 | 1 | 1 | 1 |
| **1000** | 1 | 1 | 1 | 1 | 1 | 1 |
| **10000** | 1 | 1 | 1 | 1 | 1 | 1 |

Depending on the decided target energy and an appropriate value for *λ*, this chart shows the optimal number of revolutions for which the singly-accelerating race-track machine should be constructed. Inserting bool = 0 and this optimal number of revolutions into the algorithm of the Mathematica notebook used to produce the first sample result will yield the optimal dimensions as well as the other relevant parameters studied.

Deciding the best target energy depends on the physics goals of the desired e-p collisions, which is beyond the subject of this project; however, once a target energy is chosen, studying secondary effects of dipole length and number, as well as the maximum bending field will provide some insight into orders of magnitude for λ at which the model is most effective. For example, at a machine target energy of 60 GeV, the OCS at each *λ* yields:

| | Acc/Rev | No. Revs |
|---|---|---|
| λ=1 | 1 | 4 |
| λ=10 | 1 | 3 |
| λ=100 | 1 | 2 |
| λ=1000 | 1 | 1 |
| λ=10000 | 1 | 1 |

| | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|
| λ=1 | 2.35779 | 244.161 | 260.47 | 4.07731 | 345 | 0.975308 | 2.79103 |
| λ=10 | 2.71844 | 271.885 | 329.21 | 1.91081 | 215 | 2.53188 | 1.68113 |
| λ=100 | 3.98814 | 365.086 | 486.98 | 0.609466 | 113 | 11.4156 | 0.645608 |
| λ=1000 | 7.03709 | 500.96 | 653.126 | 0.152166 | 47 | 65.1265 | 0.206494 |
| λ=10000 | 13.6238 | 830.033 | 1312.48 | 0.0482447 | 32 | 301.704 | 0.065581 |

and the OES gives:

| | Acc/Rev | No. Revs |
|---|---|---|
| λ=1 | 1 | 4 |
| λ=10 | 1 | 2 |
| λ=100 | 1 | 1 |
| λ=1000 | 1 | 1 |
| λ=10000 | 1 | 1 |

| | Circ. [km] | Tot. Cost [$$ Mill.] | Eff. Cost [$$ Mill.] | Loss/Rev [GeV] | No. Dip. | Ave. Length Dip. [m] | B [T] |
|---|---|---|---|---|---|---|---|
| λ=1 | 2.35779 | 244.161 | 260.47 | 4.07731 | 345 | 0.975308 | 2.79103 |
| λ=10 | 3.27721 | 292.815 | 326.224 | 1.67047 | 167 | 2.90254 | 1.68755 |
| λ=100 | 4.98991 | 399.393 | 446.308 | 0.469146 | 69 | 14.3859 | 0.633415 |
| λ=1000 | 7.03709 | 500.96 | 653.126 | 0.152166 | 47 | 65.1265 | 0.206494 |
| λ=10000 | 13.6238 | 830.033 | 1312.48 | 0.0482447 | 32 | 301.704 | 0.065581 |

One of the most important parameters is the maximum bending field, which needs to kept under 2T—the superconducting threshold. Both tables allow us to throw out λ = 1, and possibly λ = 10, where the superconducting region is being approached.

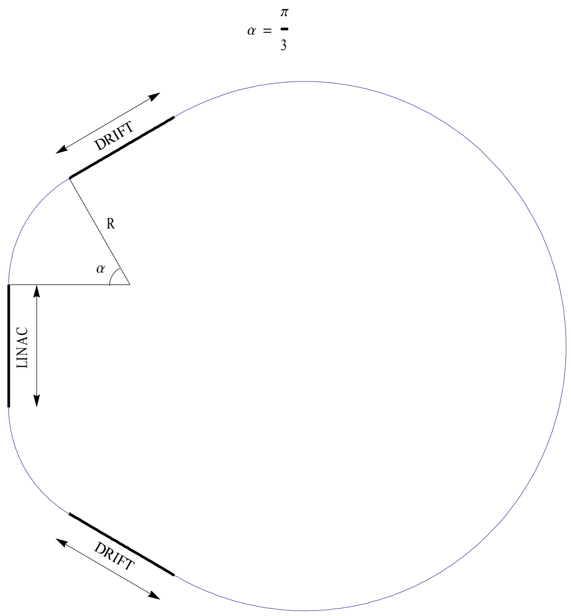
Looking then at λ = 10, the next important consideration is the number of dipoles, since total cost is realistically determined on a per dipole basis. For the OCS, the model requires at least 215 dipoles/rev at a minimizing structure of 3 revolutions, yielding 645 total dipoles. For the OES, the corresponding numbers are 167/rev and 334. At λ = 100, there is a sharp decrease, yielding a minimum of 226 total dipoles for the OCS (almost 1/3 the amount for λ = 10) and just 69 for the OES (almost 1/5 the amount for λ = 10!). In terms of just the number of dipoles needed and the average length alloted for each one (increased by a factor of almost 5 for both tables), λ = 100 is much favored over λ = 10.

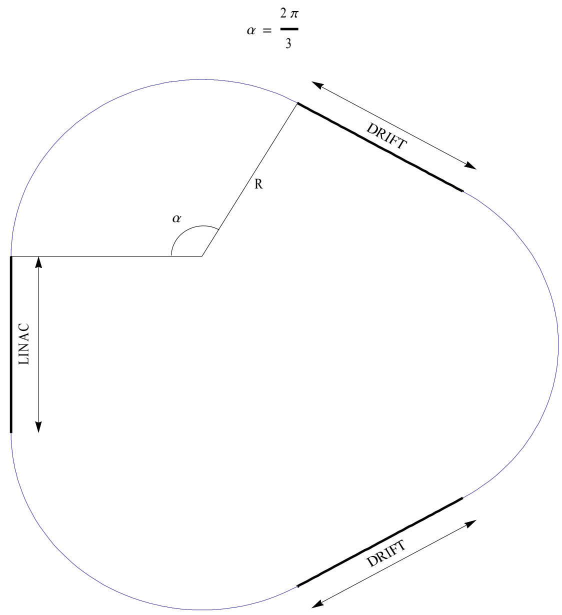
It is important to note that the minimal number of revolutions is just 1 for the last 2 λ-values for the OCS and the last 3 for the OES. At 1 revolution in the singly-accelerating structure, the $e^-$ beam undergoes just 2 accelerations, almost defeating the purpose of using a recirculating structure. Using the energy gradient of 15 MeV/m and a target energy of 60 GeV, a straight linac would require 4 km in length and cost $640 million at $160,000/m. Looking at the effective cost for λ = 1000 and λ = 10000, the model yields no advantage to using recirculation in the first place. We can throw out these 2 values of λ, leaving λ = 10 and 100.

Any λ > 10 brings down the maximum dipole field (which is already flirting with the superconducting region) and the number of dipoles needed (this realistically brings down cost of construction). Any λ ≥ 100 yields optimal effective cost at just 1 revolution, making the design and construction of a recirculating structure spatially and cost inefficient. Consequently, at 60 GeV, this model is most effective for λ values within 1 order of magnitude—between 10 and 100 (corresponding to tens of millions of dollars per GeV energy loss).

**Appendix A. Ball-Field Design**

Conceptually, and pre-computationally, the idea for the "ball-field" design was thought of as a way to bring down the radiative energy loss significantly by forcing the recirculating track of the structure out to larger radii (since synchrotron power goes like $1/R^2$, so energy loss goes approximately like 1/R). Structurally, this design comes with five parameters: the length of the linac, the length of the two drift sections, the small radius (R), the angular spread of the small circle (*α)*, and, of course, the number of revolutions. Here are two samples of the structure for 1. $\alpha < \pi/2$ and 2. $\alpha > \pi/2$.




Finding an optimal ball-field design uses a similar algorithm as the race-track design—fixing *α* and optimizing the total effective cost function of three variables (linac length, drift length, and small radius). The following samples compare, for 60 GeV, the OCS of the race-track design to the OCS of the *α = π/3* and *α = 2π/3* ball-field designs. All parameters used for the injection energy (500 MeV), the energy gradient (15 MeV/m), and cost of each kind of track (same as before) are identical for the two designs compared. As determined before, an appropriate value for λ at this energy is between 10 and 100. Referring to the OCS race-track chart, we will use N = 3 at λ = 10 and N = 2 at λ = 100.

**λ = 10**

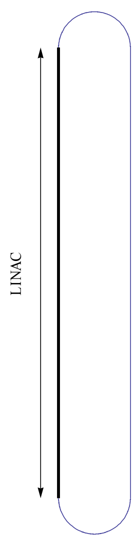
**<u>Race Track</u>**

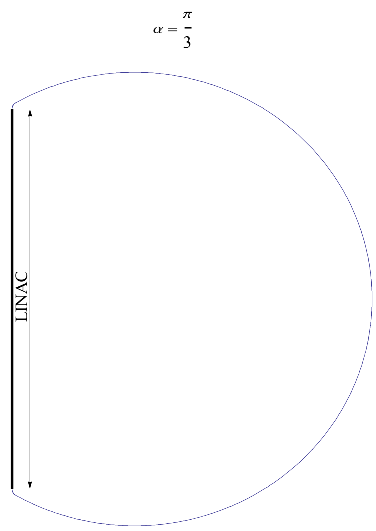
```
Optimal radius = 86.6367 m
Optimal length = 1.08704 km
Total circumference = 2.71844 km
Total energy loss to radiation = 5.73243 GeV
Total cost of design = $271.885 million
Effective cost of design = $329.21 million
```

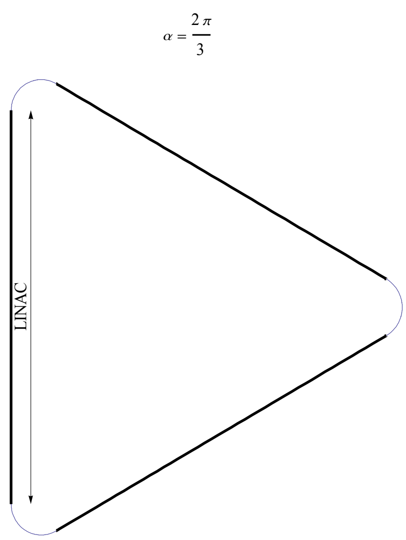
**<u>Ball Field</u>**

```
Optimal small radius = 39.0311 m
Optimal linac length = 2.00547 km
Optimal drift length = 0 km
Total circumference = 7.10074 km
Total energy loss to radiation = 0.674109 GeV
Total cost of design = $575.639 million
Effective cost of design = $ 582.38 million
```

```
Optimal small radius = 84.8961 m
Optimal linac length = 1.087 km
Optimal drift length = 1.07737 km
Total circumference = 3.7868 km
Total energy loss to radiation = 5.7299 GeV
Total cost of design = $271.839 million
Effective cost of design = $ 329.138 million
```



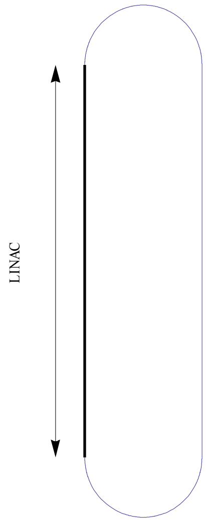



## $\lambda$ = 100

### Race Track

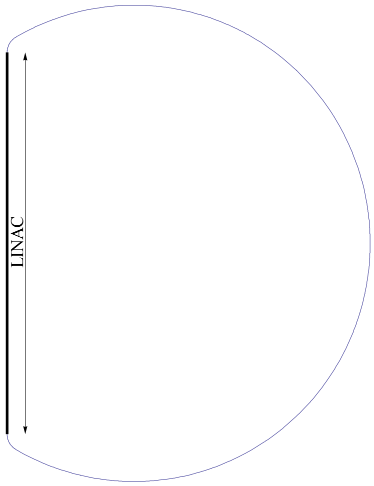
```
Optimal radius = 205.304 m
Optimal length = 1.34909 km
Total circumference = 3.98814 km
Total energy loss to radiation = 1.21893 GeV
Total cost of design = $365.086 million
Effective cost of design = $486.98 million
```




### Ball Field

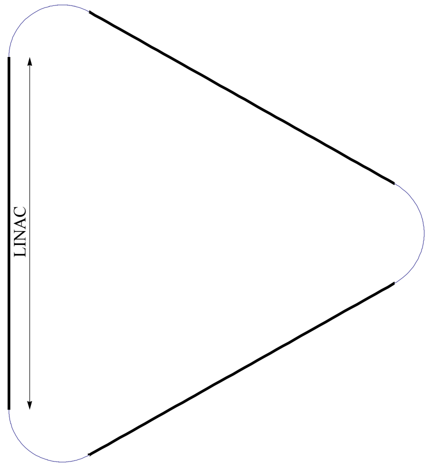
```
Optimal small radius = 93.2266 m
Optimal linac length = 1.99316 km
Optimal drift length = 0 km
Total circumference = 7.39917 km
Total energy loss to radiation = 0.304761 GeV
Total cost of design = $589.206 million
Effective cost of design = $ 619.682 million
```

$\alpha = \frac{\pi}{3}$




```
Optimal small radius = 199.215 m
Optimal linac length = 1.34906 km
Optimal drift length = 1.31418 km
Total circumference = 5.2713 km
Total energy loss to radiation = 1.21772 GeV
Total cost of design = $364.951 million
Effective cost of design = $ 486.723 million
```

$\alpha = \frac{2\pi}{3}$




As demonstrated by this one example, the $\alpha < \pi/2$ ball-field design does reduce energy loss significantly; however, it comes at about double the total cost for $\lambda$ = 10 and about 1.6 times the total cost for $\lambda$ = 100. The $\alpha > \pi/2$ ball-field design comes at very comparable costs at both $\lambda$ values, but yields a larger circumference (to be expected) and comparable energy loss—defeating the original purpose for constructing such a shape.

While this particular project explored options for the race-track design, it would definitely be worthwhile to construct a more sophisticated analysis for the ball-field design next—following a similar algorithm for the same group of ($\lambda$ , E) pairs over an incremented $\alpha$-range and comparing the results to the race-track results. Only two pairs were studied here for just two different values of $\alpha$. At this point, it would be impossible to generalize its conclusions.

### References


[1] http://www.lhec.org.uk
[2] F. Zimmermann et al, Linac-LHC ep Collider Options, EPAC'08 Genoa (2008).
[3] M. Sands, Emittance Growth From Radiation Fluctuations, SLAC/AP-047 (1985).
[4] L.C. Teng, Fermilab, TM-1269 (1984).